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STABILITY OF THE LAGRANGE SOLUTIONS OF THE RESTRICTED THREE-BODY PROBLEM FOR THE CRITICAL RATIO OF THE MASSES

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For two-dimensional and three-dimensional motions we prove the formal stability of the Lagrange solutions of the circular restricted three-body problem with a critical ratio of the masses of the primary bodies.

1. We consider the motion of three material points, mutually attracting one another according to Newton's law. The equations of motion of the problem admit particular solutions corresponding to a motion under which the three bodies form an equilateral triangle rotating in its plane about the center of mass of the three-body system. We examine the stability of these solutions for the case of the circular restricted problem in which the ratio of the masses of the primary bodies is critical.

Let the units of measurement be chosen so that the angular rate of rotation of the primary attracting bodies, the distance between them, the sum of their masses, and the constant attraction are equal to unity. In these units the mass of the smaller of the attracting bodies is equal to μ. We express the Hamiltonian function of the circular restricted three-body problem close to the triangle solution L_4 in a series [1] and we write it in the form

$$H = H_2 + H_3 + H_4 + \ldots \tag{1.1}$$

$$H_2 = \frac{1}{2}(p_1^2 + p_2^2) + q_2 p_1 - q_1 p_2 + \frac{1}{8} q_1^2 - k q_1 q_2 - \frac{5}{8} q_2^2 + \frac{1}{2}(q_3^2 + p_3^2)$$

$$H_3 = \frac{\sqrt{3}}{144}(-28kq_1^3 + 27q_1^2 q_2 + 132kq_1 q_2^2 + 27q_2^3 - 48kq_1 q_3^2 - 108 q_2 q_3^2)$$

$$H_4 = \frac{1}{384}(111q_1^4 + 400kq_1^3 q_2 - 738q_1^2 q_2^2 - 720\,kq_1 q_2^3 - 9q_2^4 +$$

$$72q_1^2 q_3^2 + 960kq_1 q_2 q_3^2 + 792q_2^2 q_3^2 - 144q_3^4), \qquad k = \frac{3\sqrt{3}}{4}(1 - 2\mu)$$

where H_m is a polynomial of degree m in the coordinates q_i and the momenta p_i, $i = 1, 2, 3$.

We consider first the case of two-dimensional motion. The frequencies ω_1 and ω_2 ($\omega_1 \geqslant \omega_2$) of an oscillating system with the Hamiltonian $H_2(q_1, q_2, p_1, p_2)$ satisfy the equation

$$\omega_4 - \omega^2 + {}^{27}/_4\, \mu(1 - \mu) = 0$$

To a first approximation we write the stability region in the form of the inequalities

$$0 < \mu < (9 - \sqrt{69}) / 18 \simeq 0.0385208\ldots \tag{1.2}$$

It was shown in [2, 3] that the Lagrange solutions of the two-dimensional circular restricted three-body problem are stable for all values of μ belonging to the region (1.2) except for two values which give rise to instability. These values of μ, namely μ_1 and μ_2, correspond to resonances of the third and fourth orders

$$\mu_1 = (45 - \sqrt{1833}) / 90 \simeq 0.0242938..., \quad \omega_1 = 2\omega_2 = 2\sqrt{5} / 5$$

$$\mu_2 = (15 - \sqrt{213}) / 30 \simeq 0.0135160..., \quad \omega_1 = 3\omega_2 = 3\sqrt{10} / 10$$

As yet unanswered is the question of stability of the solutions for the boundary values μ of the region (1.2). When $\mu = 0$, the question is easily answered since the problem reduces to investigating the stability of motion of a material point around a fixed center of attraction and, for such a motion, there is only orbital stability. A study of the stability when $\mu = \mu^* = (9 - \sqrt{69}) / 18$ (the critical ratio of the masses of the primary bodies) involves a difficulty, namely, that for this value of μ the frequencies of the linear problem are equal and the linear system is unstable. We examine the stability of the Lagrange solutions for $\mu = \mu^*$ in Sect. 2.

The stability of the Lagrange solutions in the three-dimensional case was investigated in [1, 4]. As in the two-dimensional case, when $\mu = \mu^*$ no conclusions have as yet been reached concerning the stability of the Lagrange solutions. We consider this question in Sect. 3.

2. We investigate now the problem of the stability of the Lagrange solutions of the two-dimensional problem with a critial ratio of the masses of the primary bodies ($\mu = \mu^*$).

The stability of an equilibrium position of an autonomous Hamiltonian system with two degrees of freedom was investigated in [5] for the case of equal frequencies ($\omega_1 = \omega_2 = \omega$) of the linear problem. The cases of prime and nonprime elementary divisors of the characteristic matrix were studied. In [5] a real normal form of the linear problem was obtained in the case of nonprime elementary divisors, and a constructive way for obtaining a normalizing linear canonical transformation N, was indicated. It was also shown that using a nonlinear canonical change of variables, the Hamiltonian function can be reduced to the form

$$H = {}^1/_2 (Q_1^2 + Q_2^2) + \omega (Q_1P_2 - Q_2P_1) + \qquad (2.1)$$
$$(P_1^2 + P_2^2) [A (P_1^2 + P_2^2) + B (Q_1P_2 - Q_2P_1) + C (Q_1^2 + Q_2^2)] + H_5 + \ldots$$

and it was proved that for $A > 0$ the equilibrium position $Q_1 = Q_2 = P_1 = P_2 = 0$, has formal stability [6], while for $A < 0$ it is unstable in the Liapunov sense.

In the problem of the stability of the Lagrange solutions for $\mu = \mu^*$ the characteristic matrix of the linear system has nonprime elementary divisors and its eigenvalues are equal to $\pm i \sqrt{2} / 2$. The linear real canonical transformation $\mathbf{q} = N\mathbf{q}^*$, where

$$\mathbf{q} = \begin{Vmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{Vmatrix}, \quad N = \begin{Vmatrix} \frac{43\sqrt{10}}{100} & -\frac{\sqrt{115}}{50} & -\frac{\sqrt{230}}{10} & -\frac{3\sqrt{5}}{10} \\ -\frac{\sqrt{230}}{100} & \frac{\sqrt{5}}{50} & -\frac{\sqrt{10}}{10} & \frac{\sqrt{115}}{10} \\ \frac{\sqrt{230}}{100} & -\frac{3\sqrt{5}}{20} & -\frac{\sqrt{10}}{4} & 0 \\ \frac{17\sqrt{10}}{50} & -\frac{11\sqrt{115}}{100} & -\frac{\sqrt{230}}{20} & -\frac{2\sqrt{5}}{5} \end{Vmatrix}, \quad \mathbf{q}^* = \begin{Vmatrix} q_1^* \\ q_2^* \\ p_1^* \\ p_2^* \end{Vmatrix}$$

reduces the quadratic part of the Hamiltonian function for the two-dimensional circular restricted three-body problem to a normal form. Making all subsequent calculations according to formulas given in [5], we can reduce the Hamiltonian function to the form (2.1) with $A = 0.603... > 0$. Consequently, for the critical ratio of the masses of the primary bodies the Lagrange solutions of the two-dimensional circular restricted three-body problem are formally stable.

3. We now consider the three-dimensional problem for the critical ratio of the masses. Here $\omega_1 = \omega_2 = \sqrt{2}/2$, $\omega_3 = 1$. By means of a linear real canonical transformation we reduce the quadratic part H_2 of the Hamiltonian function (1.1) to a real normal form. In doing this we transform the variables of the two-dimensional motion using the matrix N of Sect. 2, and we leave the variables q_3 and p_3 unchanged. The Hamiltonian function (1.1) then assumes the form

$$H = \frac{1}{2}(q_1^{*2} + q_2^{*2}) + \frac{\sqrt{2}}{2}(q_1^* p_2^* - q_2^* p_1^*) + \frac{1}{2}(q_3^2 + p_3^2) + \sum_{\nu=3}^{\infty} h_{\nu_1\nu_2\nu_3\nu_4\nu_5} q_1^{*\nu_1} q_2^{*\nu_2} p_1^{*\nu_3} p_2^{*\nu_4} q_3^{\nu_5}, \quad \nu = \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 \tag{3.1}$$

where the coefficients, to be used in what follows, are

$$h_{10002} = -\frac{\sqrt{3}}{12}(\sqrt{23}n_{11} + 9n_{21}), \quad h_{01002} = -\frac{\sqrt{3}}{12}(\sqrt{23}\,n_{12} + 9n_{22})$$

$$h_{00102} = -\frac{\sqrt{3}}{12}(\sqrt{23}\,n_{13} + 9n_{23}), \quad h_{00012} = -\frac{\sqrt{3}}{12}(\sqrt{23}\,n_{14} + 9n_{24})$$

$$h_{00004} = -\frac{3}{8}$$

As was done in [5], we can, by applying Birkhoff's transformation [7], completely annihilate the terms of the third order in the coordinates and momenta in Eq. (3.1); in addition, we can simplify the terms of the fourth order. We reduce the Hamiltonian function (3.1) to the form

$$\begin{aligned} H = {} & {}^1/_2 (Q_1^2 + Q_2^2) + \sqrt{2}/2\,(Q_1P_2 - Q_2P_1) + {}^1/_2 (Q_3^2 + P_3^2) + \\ & (P_1^2 + P_2^2)\,[A\,(P_1^2 + P_2^2) + B\,(Q_1P_2 - Q_2P_1) + C\,(Q_1^2 + Q_2^2)\,] + \\ & (Q_3^2 + P_3^2)\,[D\,(P_1^2 + P_2^2) + E\,(Q_1P_2 - Q_2P_1) + F\,(Q_3^2 + P_3^2)] + H_5 + \ldots \end{aligned}$$

$$\begin{aligned} F = {} & \left[\frac{1}{2(2-\omega)} - \frac{1}{2(2+\omega)} + \frac{2}{\omega}\right][h_{10002}h_{00102} - h_{01002}h_{00012}] + \\ & \left[\frac{1}{4(2-\omega)^2} + \frac{1}{4(2+\omega)^2} + \frac{1}{\omega^2}\right][h^2_{00102} + h^2_{00012}] + \frac{3}{8}h_{00004} \end{aligned}$$

Substituting numerical values, we obtain $F = 9.660...$, while the coefficient A was calculated in Sect. 2.

We now prove the formal stability of the Lagrange solutions in the three-dimensional case. We can show, with the aid of an infinite number of steps of Birkhoff's transformation (possibly diverging), that the Hamiltonian function reduces to the form

$$H = \frac{1}{2}(Q_1^2 + Q_2^2) + \frac{\sqrt{2}}{2}(Q_1P_2 - Q_2P_1) + \frac{1}{2}(Q_3^2 + P_3^2) + \sum_{\alpha=2}^{\infty} a_{\alpha_1\alpha_2\alpha_3\alpha_4}(Q_1^2 + Q_2^2)^{\alpha_1}(P_1^2 + P_2^2)^{\alpha_2}(Q_1P_2 - Q_2P_1)^{\alpha_3}(Q_3^2 + P_3^2)^{\alpha_4} \tag{3.2}$$

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

where the α_i are nonnegative integers.

It can be readily verified that a canonical system with the Hamiltonian (3.2) has three formal integrals

$$H = \text{const}, \quad (Q_1P_2 - Q_2P_1) = \text{const}, \quad (Q_3^2 + P_3^2) = \text{const}$$

Consequently, the function

$$G \equiv H - \sqrt{2}/2\,(Q_1P_2 - Q_2P_1) - {}^1/_2\,(Q_3^2 + P_3^2)$$

is also a formal integral of the system with the Hamiltonian (3.2).

In the expansion $G = G_2 + G_4 + G_6 + \ldots$ the quantity

$$\begin{aligned} G_2 + G_4 = {}^1/_2\,(Q_1^2 + Q_2^2) + A\,(P_1^2 + P_2^2)^2 + F\,(Q_3^2 + P_3^2)^2 + \\ (P_1^2 + P_2^2)\,[B\,(Q_1P_2 - Q_2P_1) + C\,(Q_1^2 + Q_2^2)] + \\ (Q_3^2 + P_3^2)[D\,(P_1^2 + P_2^2) + E\,(Q_1P_2 - Q_2P_1)] \end{aligned}$$

is a positive definite function of its variables for $A > 0$ and $F > 0$. Consequently, the formal stability of the Lagrange solutions of the three-dimensional circular restricted three-body problem with a critical ratio of the masses of the primary bodies follows from the above reasoning [6].

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